

# The freedom to choose neutron star magnetic field equilibria

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## ABSTRACT

Our ability to interpret and glean useful information from the large body of observations of strongly magnetised neutron stars rests largely on our theoretical understanding of magnetic field equilibria. We answer the following question: is one free to arbitrarily prescribe magnetic equilibria such that fluid degrees of freedom can balance the equilibrium equations? We examine this question for various models for neutron star matter; from the simplest single-fluid barotrope to more realistic non-barotropic multifluid models with superfluid/superconducting components, muons and entropy. We do this for both axi- and non-axisymmetric equilibria, and in Newtonian gravity and general relativity. We show that, in axisymmetry, the most realistic model allows complete freedom in choosing a magnetic field equilibrium whereas non-axisymmetric equilibria are never completely arbitrary.

**Key words:** stars: neutron – stars: magnetic fields

## 1 INTRODUCTION

The structure of magnetic fields in the interior of neutron stars has undergone intense scrutiny in recent years. The endeavour has been mostly motivated by X-ray observations of strongly magnetised neutron stars. For example, bursts and giant flares occurring in magnetars are commonly believed to be powered by these objects’ super-strong magnetic fields (see Woods & Thompson 2006; Mereghetti et al. 2015, for reviews). Other isolated neutron stars—colloquially known by the sobriquet ‘magnificent seven’—are kept warm by their evolving magnetic fields and emit intense thermal radiation (Haberl 2007). In fact, the resulting theory-observations synergy appears to converge to a notion of a magnetic field-dominated evolutionary link between various sub-families of the neutron star population (Kaspi 2010; Viganò et al. 2013).

For the neutron star theorist wishing to delve into the physics of neutron star magnetic fields, a reasonable starting point is determining the nature of magnetic field equilibria. There is a significant corpus of recent work on that topic, both analytical (e.g., Haskell et al. 2008; Reisenegger 2009; Cioffi et al. 2009; Mastrano et al. 2011; Glampedakis et al. 2012; Lasky & Melatos 2013; Cioffi & Rezzolla 2013) and numerical (e.g., Braithwaite 2007, 2009; Lasky et al. 2011; Kiuchi et al. 2011;

Cioffi & Rezzolla 2012; Lander & Jones 2012; Lander 2013; Fujisawa & Eriguchi 2013; Palapanidis et al. 2015; Bucciantini et al. 2015). One important result that has emerged is that magnetohydrodynamic (MHD) equilibria are greatly influenced by the properties of matter or, in other words, by the equation of state (EOS). The key physics factor is that of *barotropic* versus *non-barotropic* matter, the latter allowing a much larger family of magnetic field equilibria (Reisenegger 2009). A closely related factor is that of the imposed *symmetries* in the system: for practical reasons most existing work assumes axisymmetric (2-D) magnetic equilibria; very little is known of non-axisymmetric (3-D) equilibria.

This paper addresses a more ‘global’ issue: *to what extent MHD equilibria in neutron stars are arbitrary for given EOSs, with or without axisymmetry*. Here, ‘arbitrary’ means that a magnetic field  $\mathbf{B}$  can be freely prescribed (assuming it obeys  $\nabla \cdot \mathbf{B} = 0$ ), knowing that the available fluid degrees of freedom can satisfy the equilibrium equations.

In order to address this crucial issue we start from the simplest case of a single-fluid barotropic model and subsequently consider more realistic neutron star models with stratification, superfluidity, departure from chemical equilibrium, muons and entropy. For each case we study both axisymmetric and non-axisymmetric equilibria. In some cases we find that the magnetic field *cannot* be freely prescribed, and is required to solve a Grad-Shafranov differential equation (i.e. the MHD equilibrium equation for the magnetic scalar ‘stream functions’ and the fluid parameters of the

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**Table 1.** Overview of results. For each type of equation of state, symmetry of the magnetic field and type of gravity (i.e., Newtonian/general relativistic), we calculate whether an arbitrary magnetic field can be balanced by the degrees of freedom in the fluid, and hence whether one is arbitrarily free to prescribe a magnetic field. The final column of the table lists the section of the paper in which the relevant calculations can be found.

Equation of state	Symmetry	Gravity	Free to prescribe?	Section
Single fluid, barotropic	axi- & non-axisymmetric	Newtonian	✗	2.1
Single fluid, non-barotropic	axisymmetric	Newtonian	✓	2.2.1
Single fluid, non-barotropic	non-axisymmetric	Newtonian	✗	2.2.2
Single fluid, elastic crust	axi- & non-axisymmetric	Newtonian	✓	2.3
Multifluid, cold <i>npe</i> matter	axi- & non-axisymmetric	Newtonian	✗	3.1
Multifluid, hot <i>npeμ</i> matter	axisymmetric	Newtonian	✓	3.2.1
Multifluid, hot <i>npeμ</i> matter	non-axisymmetric	Newtonian	✗	3.2.2
Single fluid, barotropic	axisymmetric	general relativistic	✗	4.2
Single fluid, non-barotropic	axisymmetric	general relativistic	✓	4.3

non-magnetic star) while in other cases the field can be completely ‘user-specified’. We also find that non-axisymmetry entails additional constraints for the magnetic field. A summary of our main results can be found below in Section 1.1.

The paper is set out as follows: in §2 we develop the general formalism for a single fluid star, applying this specifically to barotropic and stratified matter in §2.1 and §2.2, respectively. We apply specific examples of axi- and non-axisymmetric magnetic field equilibria in §2.2.1 and §2.2.2, respectively. In §2.3 we digress and discuss magnetic field equilibrium in neutron star crusts. In §3 we generalise the formalism to multifluid neutron stars with superfluid/superconducting components. We first treat the case of cold *npe* matter and subsequently move to the most realistic model considered in this paper, accounting for the presence of muons and a finite temperature. Finally, in §4 we consider full general relativistic (GR) MHD equilibria with both barotropic and non-barotropic EOS.

### 1.1 Overview of results

This section provides a compact summary of our main results, having in mind the ‘fast-track’ reader who may not have a penchant for detailed calculations and numerous equations. Our findings are shown in Table 1.

We can categorise our results in the following way:

(i) assuming axisymmetry, we first recover the well-known results for single-fluid barotropic stars (the field is required to solve a Grad-Shafranov equation) and single-fluid non-barotropic stars (the field can be freely prescribed). Moving to multifluid neutron stars (which are always non-barotropic) with cold *npe* matter, we find that the system effectively behaves as a barotrope with an accompanying Grad-Shafranov equation. The addition of muons and entropy in the previous model (hot *npeμ* matter) restores the complete freedom in prescribing an MHD equilibrium.

(ii) non-axisymmetric equilibria in the core are *never* completely arbitrary, even when the magnetic field is not required to solve a Grad-Shafranov equation. This is due to the presence of an additional constraint equation between the azimuthal and non-azimuthal magnetic force components.

(iii) the transition from Newtonian MHD to GRMHD does not produce any qualitative changes and the conclu-

sions of (i) and (ii) remain valid (this why we show just two cases of GRMHD equilibria).

(iv) The magnetic equilibrium in the crust can be freely specified provided we allow for a strained crust and the associated elastic force.

## 2 SINGLE FLUID STARS

The starting point for considering MHD equilibria in neutron stars is that of a ‘single-fluid’ model. This term is actually a misnomer because we actually account for the presence of various constituents (such as neutrons, protons and electrons). It is, however, assumed that these particles (modulo the much less massive electrons) always move in unison (on hydrodynamical scales) as a result of their mutual collisions.

Assuming for simplicity a non-rotating static system (adding rotation is trivial as it amounts to adding a centrifugal term in the gravitational potential), the single fluid MHD equilibrium is described by the Euler equation

$$\nabla p + \rho \nabla \Phi = \mathbf{F}_L, \quad (1)$$

where  $p$ ,  $\rho$ , and  $\Phi$  are respectively the pressure, density and gravitational potential, and  $\mathbf{F}_L$  is the Lorentz force given by

$$\mathbf{F}_L = \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B}. \quad (2)$$

In this paper, we treat the magnetic force as a perturbation on some spherical background (an approximation well justified for all astrophysical magnetic fields), such that

$$\nabla p + \rho \nabla \Phi = 0, \quad (3)$$

$$\nabla \delta p + \delta \rho \nabla \Phi + \rho \nabla \delta \Phi = \mathbf{F}_L. \quad (4)$$

Here, Eqn. (3) is the background hydrostatic equilibrium, and Eqn. (4) is the leading-order perturbation equation, where a  $\delta$  denotes perturbed quantities and second-order terms have been neglected. Combining (3) and (4) implies

$$\nabla (\delta h + \delta \Phi) + \frac{1}{\rho} (\delta p \nabla \rho - \delta \rho \nabla p) = \mathbf{F}_L, \quad (5)$$

where  $\delta h \equiv \delta p / \rho$  is the enthalpy perturbation.

The possible geometry of a magnetic field depends on the matter content of the star through the EOS. In general the latter will have the functional form  $p = p(\rho, x)$ , where

$x$  is typically identified with the neutron star composition (i.e., the proton fraction,  $x_p$ ). One typically defines a pair of adiabatic indices according to

$$\nabla p = \frac{p}{\rho} \Gamma_0 \nabla \rho, \quad (6)$$

$$\Gamma_1 = \frac{\rho}{p} \frac{\partial p}{\partial \rho} \bigg|_x. \quad (7)$$

Expanding the left hand side of Eqn. (6) and substituting Eqn. (7) leads to

$$\frac{p}{\rho} (\Gamma_0 - \Gamma_1) \nabla \rho = \frac{\partial p}{\partial x} \nabla x. \quad (8)$$

Importantly, this equation implies that  $\Gamma_0 = \Gamma_1$  if and only if there is no stratification in the star.

After some straightforward manipulations we can write the Euler equation (5) in the equivalent form,

$$\nabla (\delta h + \delta \Phi) + \Lambda_\rho \nabla \rho = \frac{1}{\rho} \mathbf{F}_L, \quad (9)$$

where

$$\Lambda_\rho \equiv \frac{p}{\rho^3} \left[ (\Gamma_1 - \Gamma_0) \delta \rho + \frac{\rho}{p} \frac{\partial p}{\partial x} \delta x \right]. \quad (10)$$

We can rewrite  $\Lambda_\rho$  without  $\delta x$  appearing explicitly. From the EOS we have

$$\delta h = \frac{p \Gamma_1}{\rho^2} \delta \rho + \frac{\partial p}{\partial x} \frac{\delta x}{\rho}, \quad (11)$$

implying

$$\Lambda_\rho = \frac{1}{\rho} \left( \delta h - \frac{p \Gamma_0}{\rho^2} \delta \rho \right). \quad (12)$$

In order to proceed further we need to make a choice for the EOS. In the following sections we consider separately the cases of barotropic and non-barotropic matter.

## 2.1 Barotropic matter

We begin by considering barotropic matter in which the EOS is simply  $p = p(\rho)$ , which implies no stratification. In this case,  $\Lambda_\rho = 0$  and taking the curl of Eqn. (9) results in

$$\nabla \times \left( \frac{1}{\rho} \mathbf{F}_L \right) = 0, \quad (13)$$

which is an equation involving only the magnetic-field degrees of freedom and the *background* density.

If in addition the system is assumed to be *axisymmetric*, we can decompose the field into poloidal and toroidal components,  $\mathbf{B} = \mathbf{B}_P + \mathbf{B}_T$ , where

$$\mathbf{B}_P = \nabla \Psi \times \nabla \varphi, \quad \mathbf{B}_T = T \nabla \varphi, \quad (14)$$

and the two degrees of freedom are represented by the scalar stream functions  $\Psi(r, \theta)$  and  $T(r, \theta)$  (here we use standard spherical coordinates). In axisymmetry, the azimuthal component of the Lorentz force must vanish,  $F_L^\varphi = 0$ , and one can easily show that  $\nabla \Psi \times \nabla T = 0 \Rightarrow T = T(\Psi)$ . The full Lorentz force can be written as

$$\mathbf{F}_L = -\frac{1}{4\pi\varpi^2} (\Delta_* \Psi + TT') \nabla \Psi \equiv \mathcal{A} \nabla \Psi, \quad (15)$$

where a prime denotes a derivative with respect to the argument,  $T' = dT/d\Psi$ ,  $\varpi = r \sin \theta$ , and

$$\Delta_* \Psi = \nabla^2 \Psi - \frac{2}{r} \left[ \hat{\mathbf{r}} \cdot \nabla \Psi + \cot \theta (\hat{\boldsymbol{\theta}} \cdot \nabla \Psi) \right], \quad (16)$$

where  $\hat{\mathbf{r}}$  and  $\hat{\boldsymbol{\theta}}$  are unit vectors.

The so-called Grad-Shafranov equation can be derived from Eqn. (13)

$$\frac{\mathcal{A}}{\rho} = M(\Psi) \Rightarrow \Delta_* \Psi + TT' = 4\pi\varpi^2 \rho M, \quad (17)$$

where  $M(\Psi)$  is an arbitrary function.

Equation (17) implies that axisymmetric magnetic field equilibria in barotropic stars come from a restricted class of solutions. That is, a single equation governs both the poloidal and toroidal component, implying one is not free to arbitrarily choose both components independently.

In general, once  $\mathbf{B}$  has been obtained from Eqn. (17), the perturbed fluid's degrees of freedom can be found from Eqn. (9) with  $\Lambda_\rho = 0$ . We have,

$$\nabla (\delta h + \delta \Phi) = M \nabla \Psi. \quad (18)$$

This is solved by,

$$\delta h + \delta \Phi = g(\Psi) = \int d\Psi M(\Psi). \quad (19)$$

The system is closed with the perturbed Poisson equation  $\nabla^2 \delta \Phi = 4\pi G \delta \rho$ .

Next, we consider 3-D *non-axisymmetric* magnetic field equilibria in barotropic matter. Although perhaps not so well known in the literature, it is still possible to parametrise  $\mathbf{B}$  in terms of a pair of scalar functions. These are the so-called Euler potentials  $\{\alpha(r, \theta, \varphi), \beta(r, \theta, \varphi)\}$  and are defined<sup>1</sup> as (e.g. see [Yahalom & Lynden-Bell 2008](#)):

$$\mathbf{B} = \nabla \alpha \times \nabla \beta, \quad (20)$$

and as a consequence  $\nabla \cdot \mathbf{B} = 0$  is trivially satisfied.

In terms of the Euler potentials, the electric current and the Lorentz force become,

$$\mathbf{J} = \frac{c}{4\pi} [\nabla^2 \beta \nabla \alpha - \nabla^2 \alpha \nabla \beta + (\nabla \beta \cdot \nabla) \nabla \alpha - (\nabla \alpha \cdot \nabla) \nabla \beta], \quad (21)$$

$$\mathbf{F}_L = h_{\alpha\beta} \nabla \alpha + h_{\beta\alpha} \nabla \beta, \quad (22)$$

where

$$\begin{aligned} h_{\alpha\beta} &\equiv \frac{1}{4\pi} \mathbf{J} \cdot \nabla \beta \\ &= \frac{1}{4\pi} [\nabla^2 \beta (\nabla \alpha \cdot \nabla \beta) - \nabla^2 \alpha (\nabla \beta \cdot \nabla \beta) \\ &\quad + \nabla \beta \cdot (\nabla \beta \cdot \nabla) \nabla \alpha - \nabla \beta \cdot (\nabla \alpha \cdot \nabla) \nabla \beta]. \end{aligned} \quad (23)$$

Based on these results we attempt to derive a ‘‘Grad-Shafranov’’ equation for a general non-axisymmetric equilibrium. From (13) we have,

$$\nabla \left( \frac{h_{\alpha\beta}}{\rho} \right) \times \nabla \alpha + \nabla \left( \frac{h_{\beta\alpha}}{\rho} \right) \times \nabla \beta = 0. \quad (24)$$

This is solved by

$$h_{\alpha\beta} = \rho \mathcal{M}(\alpha), \quad h_{\beta\alpha} = \rho \tilde{\mathcal{M}}(\beta), \quad (25)$$

where  $\mathcal{M}, \tilde{\mathcal{M}}$  are arbitrary functions (not necessarily different). Equations (25) can be viewed as the non-axisymmetric generalisation of the 2-D Grad-Shafranov equation (17).

<sup>1</sup> We note that the Euler potentials are not uniquely defined. For instance, (20) is invariant with respect to the gauge transformation  $\beta \rightarrow \beta + g(\alpha)$ , where  $g$  is arbitrary.

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To conclude this section, it is interesting to investigate the connection between the Euler potentials  $\alpha, \beta$  and the stream functions  $\Psi, T$  in axisymmetry. We have,

$$\mathbf{B} = \nabla \Psi \times \nabla \varphi + T \nabla \varphi = \nabla \alpha + \nabla \beta. \quad (26)$$

The inner and cross product with  $\nabla \varphi$  leads to, respectively,

$$T = \varpi(\dot{\varphi} \times \nabla \alpha) \cdot \nabla \beta, \quad (27)$$

$$\nabla \Psi = \partial_\varphi \beta \nabla \alpha - \partial_\varphi \alpha \nabla \beta. \quad (28)$$

At this point we can choose either  $\partial_\varphi \alpha = 0$  or  $\partial_\varphi \beta = 0$ . Opting for the former,

$$\nabla \Psi \times \nabla \alpha = 0 \Rightarrow \Psi = \Psi(\alpha), \quad (29)$$

which shows that the  $\alpha$  and  $\Psi$  surfaces coincide. Also,

$$\partial_\varphi \beta = \Psi' \Rightarrow \beta = \Psi' \varphi + K(r, \theta), \quad (30)$$

where  $K$  is an integration ‘constant’ and

$$T = \varpi(\dot{\varphi} \times \nabla \alpha) \cdot \nabla K. \quad (31)$$

Equations (29) and (31) are the desired relations between the Euler potentials and the stream functions in axisymmetry.

### 2.2 Stratified matter

In contrast to the barotropic case, we show in this section that there is significantly more freedom in the stratified case, in the sense that any user-specified magnetic field can be in equilibrium.

For cold (i.e. zero temperature) neutron star matter, the appropriate second parameter in the EOS is the proton fraction, i.e.  $x = x_p$ . For simplicity, we adopt the Cowling approximation; note that this is not restrictive since  $\delta\Phi$  comes with the perturbed Poisson equation. The relevant Euler equation is:

$$\nabla \delta h + \Lambda_\rho \nabla \rho = \frac{1}{\rho} \mathbf{F}_L, \quad (32)$$

with  $\Lambda_\rho$  given by (12).

#### 2.2.1 Axisymmetric equilibria

In axisymmetry, we again decompose the field into poloidal and toroidal components, and again have  $T = T(\Psi)$  for the magnetic stream functions and  $\mathbf{F}_L = \mathcal{A} \nabla \Psi$ . However, taking the curl of both sides of Eqn. (32) implies

$$\begin{aligned} \nabla \left( \frac{\mathcal{A}}{\rho} \right) \times \nabla \Psi &= \nabla \Lambda_\rho \times \nabla \rho = \\ &= \frac{\rho'}{\rho} \left[ \hat{\mathbf{r}} \times \left\{ \frac{p\Gamma_0}{\rho^2} \nabla \delta \rho - \nabla \delta h \right\} \right] \neq 0 \Rightarrow \frac{\mathcal{A}}{\rho} \neq M(\Psi), \end{aligned} \quad (33)$$

where a prime in a background quantity denotes a radial derivative. The above equation means that there is no Grad-Shafranov equation. In other words, there is no single equation that governs both the toroidal and poloidal field components, implying these components can be specified independently from one another. Faced with this result, we simply consider  $\mathbf{B}$  as given and investigate if the fluid parameters can be adjusted to ensure an MHD equilibrium.

Unlike the barotropic case, the perturbations  $\delta h$  and

$\delta \rho$  are not required to be linearly dependent (the remaining function  $\delta x_p$  follows trivially from the EOS relation once  $\delta h$  and  $\delta \rho$  are known, see Eqn. (11)).

Expanding the Euler equation (32) in components:

$$\partial_r \delta p - \frac{p\Gamma_0 \rho'}{\rho^2} \delta \rho = \mathcal{A} \partial_r \Psi, \quad (34)$$

$$\partial_\theta \delta p = \mathcal{A} \partial_\theta \Psi. \quad (35)$$

Integrating the bottom equation

$$\delta p = \zeta(r) + \int d\theta \mathcal{A}(r, \theta) \partial_\theta \Psi(r, \theta) \quad (36)$$

where  $\zeta(r)$  is an arbitrary spherical ‘gauge’ function. Finally, putting this back into Eqn. (34) implies

$$\delta \rho = \frac{\rho^2}{p\Gamma_0 \rho'} \left[ \zeta' - \mathcal{A} \partial_r \Psi + \partial_r \left\{ \int d\theta \mathcal{A}(r, \theta) \partial_\theta \Psi(r, \theta) \right\} \right]. \quad (37)$$

These equations show that there is sufficient freedom in the fluid variables to balance a given magnetic field. That is, for a given  $\mathbf{B}$ , once one writes down the functional form for the poloidal and toroidal field, the Euler equations can be fully solved using Eqns. (36) and (37) for the perturbed density and pressure such that the system is in MHD equilibrium.

#### 2.2.2 Non-axisymmetric equilibria

We again consider the equilibrium described by the Euler equation (32) but this time without the assumption of axisymmetry – this means that we now have three equations for the two unknown functions  $\delta h$  and  $\delta \rho$ . As a result, we expect to have a non-trivial relation between the magnetic force components. Let us see how this works in detail.

Decomposing (32) in its components:

$$\partial_r \delta h + \frac{\rho'}{\rho} \left( \delta h - \frac{p\Gamma_0}{\rho^2} \delta \rho \right) = \frac{1}{\rho} F_L^r, \quad (38)$$

$$\Rightarrow \partial_r \delta p - \frac{p\Gamma_0 \rho'}{\rho^2} \delta \rho = F_L^r, \quad (39)$$

$$\partial_\theta \delta h = \frac{r}{\rho} F_L^\theta \Rightarrow \partial_\theta \delta p = r F_L^\theta, \quad (40)$$

$$\partial_\varphi \delta h = \frac{\varpi}{\rho} F_L^\varphi \Rightarrow \partial_\varphi \delta p = \varpi F_L^\varphi. \quad (41)$$

The radial equation is the only one featuring  $\delta \rho$  and therefore we solve it with respect to that parameter:

$$\delta \rho = \frac{\rho^2}{p\Gamma_0 \rho'} (\partial_r \delta p - F_L^r). \quad (42)$$

We are left with two equations for  $\delta p$ . It is easy to combine them, eliminate  $\delta p$ , and arrive to a condition for the magnetic force components:

$$\partial_\varphi F_L^\theta = \partial_\theta (\sin \theta F_L^\varphi). \quad (43)$$

We can subsequently use either of (40) or (41) to obtain  $\delta p$ . Using the former,

$$\delta p = \zeta(r, \varphi) + r \int d\theta F_L^\theta \quad (44)$$

where  $\zeta$  is an arbitrary function.

The condition (43) implies that the non-axisymmetric system does not allow an arbitrarily specified  $\mathbf{B}$  field. It should be emphasized that this condition is valid for both



stratified and barotropic matter (since both cases share the same  $\theta$  and  $\varphi$  Euler components). Clearly, in the limit of an axisymmetric field, it is trivially satisfied.

We can further expand (43) with the help of the Euler potentials  $\alpha, \beta$ . Using the Lorentz force result (22) we find,

$$\partial_\varphi h_{\alpha\beta} \partial_\theta \alpha + \partial_\varphi h_{\beta\alpha} \partial_\theta \beta = \partial_\theta h_{\alpha\beta} \partial_\varphi \alpha + \partial_\theta h_{\beta\alpha} \partial_\varphi \beta. \quad (45)$$

This is solved by

$$h_{\alpha\beta} = f(r)\mathcal{M}(\alpha), \quad h_{\beta\alpha} = \tilde{f}(r)\tilde{\mathcal{M}}(\beta), \quad (46)$$

where all functions are arbitrary. Interestingly, this class of solutions also encompasses the solution (25) of the 3-D ‘Grad-Shafranov’ equation (24) we derived for barotropic matter. In other words, for a barotropic non-axisymmetric system, (25) simultaneously solves (24) and (45). On the other hand, the 3-D stratified system is only constrained by (45) and allows a broader family of solutions.

### 2.2.3 Case study: tilted torus magnetic field

To the best of our knowledge, there is only one analytic, non-axisymmetric magnetic field equilibrium in the literature: the so-called tilted torus magnetic fields (Lasky & Melatos 2013). These fields are physically motivated toy models for fields in proto-neutron stars. Differential rotation or  $r$ -mode instabilities could wind up a strong toroidal component in the stellar core, whose axis of symmetry is aligned with the rotation axis of the star. There is no a priori reason for the progenitor’s field to be aligned with the remnant’s rotation axis, implying the star is likely to have two misaligned components. Tilted-torus models posit that this misaligned field is a ‘purely poloidal’ field, whose axis of symmetry is misaligned with the rotation axis, and hence misaligned with the axis of symmetry of the toroidal field. While this field is strictly poloidal in some inclined frame (see below), it has a non-zero azimuthal contribution when expressed in the frame whose basis is aligned with the rotation axis of the star. Lasky & Melatos (2013) also showed how tilted torus-like fields arise from GRMHD simulations of proto-neutron stars.

Mathematically, tilted torus fields are described using two coordinate systems,  $\mathbf{x}$  and  $\bar{\mathbf{x}}$ , where the barred coordinates are rotated by an angle  $\xi$  in the  $x$ - $z$  plane with respect to the unbarred coordinates. The poloidal component is expressed in the usual way in the barred frame as

$$\mathbf{B}_P = \bar{\nabla}\psi \times \bar{\nabla}\bar{\varphi}, \quad (47)$$

where  $\bar{\nabla}$  is the gradient operator in the barred coordinates,  $\bar{r} = r$  and  $\psi(r, \bar{\theta}) = f(r)\sin^2\bar{\theta}$  is the stream function. One has considerable freedom in choosing the radial function  $f(r)$  (e.g., see Mastrano et al. 2011; Akgün et al. 2013; Mastrano et al. 2013, for similar axisymmetric fields), but herein we leave this function arbitrary.

The toroidal component of the field is described in the usual way in the unbarred frame

$$\mathbf{B}_T = \mathcal{T}(\psi(r, \theta))\nabla\varphi. \quad (48)$$

In Lasky & Melatos (2013), a specific choice of  $\mathcal{T}(\psi)$  was made to satisfy integrability conditions for the force balance equations. We show below an equivalent description, but keep  $\mathcal{T}(\psi)$  general for the remainder of the derivation.

The total field is the sum of the toroidal and poloidal components, which is expressed in the unbarred frame as

$$\begin{aligned} \mathbf{B} = & \frac{2f}{r^2} (\cos\xi \cos\theta - \sin\xi \sin\theta \cos\varphi) \hat{\mathbf{r}} \\ & - \frac{f'}{r} (\cos\xi \sin\theta + \sin\xi \cos\theta \cos\varphi) \hat{\boldsymbol{\theta}} \\ & \frac{1}{r} \left( f' \sin\xi \sin\varphi + \frac{\mathcal{T}}{\sin\theta} \right) \hat{\boldsymbol{\varphi}}, \end{aligned} \quad (49)$$

where as always a prime denotes a derivative with respect to the argument.

Putting this magnetic field through Eqn. (43) we derive a condition for the magnetic field

$$f^2 f' \mathcal{T}'' \sin\xi \sin^2\theta \cos\theta \cos\varphi = 0. \quad (50)$$

In general, there are only two conditions in which this equation is generally true. Firstly, when  $\xi = 0$ , which represents the axisymmetric case, and hence equation (43) is trivially satisfied. In the non-axisymmetric case, one therefore has a condition on the toroidal field function,  $\mathcal{T}'' = d^2\mathcal{T}/d^2\psi = 0$ . In Lasky & Melatos (2013), an integrability condition was derived that implied  $\mathcal{T} = c_1(\psi - c_0)$  for  $\psi \geq c_0$  (where  $c_0, c_1$  constants) and zero elsewhere. This functional form of  $\mathcal{T}$  also satisfies the constraint equation (50).

### 2.3 Digression: magnetic equilibria in the crust

Real neutron stars are not just balls of fluid, but also have elastic crusts with magnetic fields threading both the core, crust and exterior region of the star. In this section, we show the effect of an elastic force on possible magnetic equilibria in the crust.

From an EOS perspective, the crust behaves as a barotropic layer; however one should also account for its elasticity. A crust threaded by a magnetic field is, in general, in a strained state described by a displacement field  $\boldsymbol{\xi}$  and the crustal shear modulus  $\mu$ .

MHD equilibria in this strained crust are then described by the Euler equation<sup>2</sup>,

$$\nabla(\delta h + \delta\Phi) = \frac{1}{\rho} (\mathbf{F}_L + \mathbf{F}_{el}). \quad (51)$$

Here, the Hookean force  $\mathbf{F}_{el} = \mu\nabla^2\boldsymbol{\xi}$  arises due to the crust elasticity (for simplicity we have assumed a uniform  $\mu$  and an incompressible displacement  $\nabla \cdot \boldsymbol{\xi} = 0$ , see McDermott et al. (1988) for the full expression of  $\mathbf{F}_{el}$ ). We thus have,

$$\nabla \times \left[ \frac{1}{\rho} (\mathbf{F}_L + \mathbf{F}_{el}) \right] = 0. \quad (52)$$

It does not require much meditation on Eqn. (52) to realise that the elastic force term adds an extra degree of freedom with which to balance the magnetic field. One is therefore free to prescribe any magnetic field in the neutron star crust. This freedom is readily exploited in calculations of magnetic field evolution (due to Ohmic dissipation and the Hall effect)

<sup>2</sup> Realistic neutron star crusts consist of a solid lattice and a neutron superfluid (in their inner layers) and are therefore multifluid systems. Apart from a trivial density rescaling  $\rho \rightarrow \rho_c$  in Eqn. (52), where  $\rho_c$  is the density of the lattice, this property does not alter the conclusions of this section. Moreover, in this discussion we ignore the crust’s outermost thin fluid layer (ocean).

in neutron star crusts (see for instance, [Pons & Geppert 2007](#); [Viganò et al. 2013](#); [Gourgouliatos & Cumming 2014](#)).

### 3 MULTIFLUID NEUTRON STARS

#### 3.1 Cold superfluid *npe* matter

The simplest model for superfluid neutron stars assumes *npe* matter and consists of two fluids, namely, the neutron superfluid and the combined proton-electron conglomerate. The charged particles can be counted as one fluid because they are nearly comoving under the action of the magnetic forces. Their combined equation of motion is further simplified by the negligible electron mass and overall charge neutrality. It is also expected that protons are paired and form a type II superconductor at a very early stage of a neutron star's life. Superconductivity modifies the magnetic force – this is now dominated by the tension of the quantised fluxtubes that thread the superconductor – and almost eliminates any relative proton-electron motion. To a first approximation the thermal physics of the system can be ignored since the temperature of a neutron star falls well below its Fermi temperature ( $T_F \sim 10^{12}$  K) very soon after formation.

The MHD equilibrium is now described by a pair of Euler equations (see e.g. [Glampedakis et al. 2012](#))

$$\rho_n \nabla(\tilde{\mu}_n + \Phi) = \mathbf{F}_n, \quad (53)$$

$$\rho_p \nabla(\tilde{\mu}_{pe} + \Phi) = \mathbf{F}_p, \quad (54)$$

where  $\tilde{\mu}_n = \mu_n/m$ ,  $\tilde{\mu}_{pe} = \mu_{pe}/m \equiv (\mu_p + \mu_e)/m$  are chemical potentials per unit mass (we use a common baryonic mass  $m = m_n = m_p \gg m_e$ ) and  $\mathbf{F}_n, \mathbf{F}_p$  are magnetic forces. Somewhat counterintuitively, there can be a magnetic force acting on the neutrons. This force arises when the protons are superconducting and is a consequence of the coupling between the proton fluxtubes and the neutron superfluid (see [Glampedakis et al. 2011](#), for details).

For the total pressure we have the thermodynamical relation (e.g., [Prix 2004](#))

$$\nabla p = n_n \nabla \mu_n + n_p \nabla \mu_p + n_e \nabla \mu_e = \rho_n \nabla \tilde{\mu}_n + \rho_p \nabla \tilde{\mu}_{pe}, \quad (55)$$

where  $n_n, n_p, n_e$  are particle densities (with  $n_p = n_e$  as dictated by charge neutrality).

For the non-magnetic background equilibrium we have:

$$\rho_n \nabla(\tilde{\mu}_n + \Phi) = \rho_p \nabla(\tilde{\mu}_{pe} + \Phi) = 0. \quad (56)$$

These lead to  $\tilde{\mu}_{pe} = \tilde{\mu}_n \Rightarrow \mu_p + \mu_e = \mu_n$  which is the condition for beta equilibrium. By adding the two equations we get the familiar equation for hydrostatic equilibrium  $\nabla p = -\rho \nabla \Phi$  with  $\rho = \rho_p + \rho_n$  the total density.

The equations for the perturbed magnetic system are

$$\rho_n \nabla(\delta \tilde{\mu}_n + \delta \Phi) = \mathbf{F}_n, \quad \rho_p \nabla(\delta \tilde{\mu}_{pe} + \delta \Phi) = \mathbf{F}_p. \quad (57)$$

From (55) we have for the perturbed pressure,

$$\delta p = \rho_n \delta \tilde{\mu}_n + \rho_p \delta \tilde{\mu}_{pe} \Rightarrow \delta h = (1 - x_p) \delta \tilde{\mu}_n + x_p \delta \tilde{\mu}_{pe}, \quad (58)$$

where  $x_p = \rho_p/\rho$  is the proton fraction.

The two Euler equations can be combined in a natural

way and produce an equivalent pair of a ‘total’ and ‘difference’ equations:

$$\nabla(\delta h + \delta \Phi) - \delta \beta \nabla x_p = \frac{1}{\rho} (\mathbf{F}_p + \mathbf{F}_n), \quad (59)$$

$$\rho \nabla \delta \beta = \frac{1}{x_p} \mathbf{F}_p - \frac{1}{1 - x_p} \mathbf{F}_n. \quad (60)$$

The parameter

$$\delta \beta \equiv \delta \tilde{\mu}_{pe} - \delta \tilde{\mu}_n = \frac{1}{m} (\delta \mu_p + \delta \mu_e - \delta \mu_n), \quad (61)$$

represents the departure from chemical equilibrium.

In order to proceed we need to specify the magnetic forces. For the case of normal (unpaired) protons there is only the ‘classical’ Lorentz force acting on the charged particles and we can set:

$$\mathbf{F}_n = 0, \quad \mathbf{F}_p = \mathbf{F}_L. \quad (62)$$

We note that this scenario could be the relevant one in (at least some) magnetars when the interior field exceeds the threshold ( $\sim 10^{15} - 10^{16}$  G) for the suppression of superconductivity (see [Glampedakis et al. 2011](#), for details).

The difference Euler equation (60) becomes,

$$\nabla \delta \beta = \frac{1}{\rho x_p} \mathbf{F}_L. \quad (63)$$

This obviously leads to a Grad-Shafranov equation:

$$\nabla \times \left( \frac{1}{x_p \rho} \mathbf{F}_L \right) = 0. \quad (64)$$

Apart from a rescaling  $\rho \rightarrow x_p \rho$ , this equation is identical to the one discussed earlier for single-fluid barotropic systems.

When protons are superconducting, the magnetic forces are given by (e.g., [Glampedakis et al. 2012](#)):

$$\mathbf{F}_p = \frac{1}{4\pi} [\nabla \times (H_c \hat{\mathbf{B}})] \times \mathbf{B} - \frac{\rho_p}{4\pi} \nabla \left( B \frac{\partial H_c}{\partial \rho_p} \right), \quad (65)$$

$$\mathbf{F}_n = -\frac{\rho_n}{4\pi} \nabla \left( B \frac{\partial H_c}{\partial \rho_n} \right), \quad (66)$$

where  $\hat{\mathbf{B}} = \mathbf{B}/B$  and  $H_c(\rho_p, \rho_n)$  is the lower critical field for type II superconductivity ([Tinkham 1996](#)). It is easy to show that the gradient terms in (65) and (66) can be absorbed into the chemical potentials. This rearrangement amounts to a redefinition of chemical potentials:

$$\delta \theta_n \equiv \delta \tilde{\mu}_n + \frac{B}{4\pi} \frac{\partial H_c}{\partial \rho_n}, \quad \delta \theta_p \equiv \delta \tilde{\mu}_{pe} + \frac{B}{4\pi} \frac{\partial H_c}{\partial \rho_p}. \quad (67)$$

We similarly define,

$$\delta \tilde{\beta} \equiv \delta \theta_p - \delta \theta_n = \delta \beta + \frac{B}{4\pi} \left( \frac{\partial H_c}{\partial \rho_p} - \frac{\partial H_c}{\partial \rho_n} \right), \quad (68)$$

$$\delta \tilde{h} \equiv (1 - x_p) \delta \theta_n + x_p \delta \theta_p. \quad (69)$$

In terms of the new parameters the Euler equations become:

$$\nabla(\delta \tilde{h} + \delta \Phi) - \delta \tilde{\beta} \nabla x_p = \frac{1}{\rho} \mathbf{F}_H, \quad (70)$$

$$\nabla \delta \tilde{\beta} = \frac{1}{x_p \rho} \mathbf{F}_H, \quad (71)$$

where

$$\mathbf{F}_H = \frac{1}{4\pi} [\nabla \times (H_c \hat{\mathbf{B}})] \times \mathbf{B}, \quad (72)$$

can be thought as the ‘Lorentz’ part of the superconducting force. Thus the superconducting system too admits a Grad-Shafranov-type equation, i.e.

$$\nabla \times \left( \frac{1}{x_p \rho} \mathbf{F}_H \right) = 0. \quad (73)$$

The upshot of this discussion is that, despite the non-barotropic nature of superfluid neutron stars made of cold *npe* matter, a magnetic field in equilibrium cannot be freely prescribed. In a sense, the non-barotropic degree of freedom is undone by the second fluid degree of freedom and as a result the  $\mathbf{B}$  field has to solve a Grad-Shafranov equation, much alike barotropic systems. At a qualitative level the situation bears some resemblance to that of *g*-mode oscillations: a stratified single-fluid system possesses a family of composition *g*-modes (Reisenegger & Goldreich 1992) – these modes disappear when the same system acquires a second fluid component (Prix & Rieutord 2002).

### 3.2 Hot superfluid *npeμ* matter

In this section, while we continue considering superfluid/superconducting matter, we also take into account two more properties of realistic neutron stars: the unavoidable appearance of muons (as predicted by realistic EOSs) and the finite temperature/entropy of matter. The resulting ‘hot’ *npeμ* model is the most realistic one used to date in the context of MHD equilibria in neutron stars.

The muons are expected to appear above a density threshold representative of the outer core. Once present, they participate both in the beta reactions and charge neutrality of matter. The latter property is always preserved in the MHD approximation and therefore

$$n_p = n_e + n_\mu, \quad (74)$$

is always true. For a system in beta equilibrium (as is the case for the non-magnetic background star) we should have

$$\mu_n = \mu_p + \mu_e, \quad \mu_n = \mu_p + \mu_\mu \quad \Rightarrow \quad \mu_e = \mu_\mu. \quad (75)$$

The addition of the magnetic field induces small deviations from this equilibrium and therefore we expect  $\delta\mu_{pe} \neq \delta\mu_n$  and  $\delta\mu_e \neq \delta\mu_\mu$ . At the level of hydrodynamics the muons essentially behave as ‘heavy electrons’ and are incorporated in the charged fluids conglomerate. In other words, they do not need a separate Euler equation.

In the canonical multifluid framework used in this paper (see e.g., Prix 2004) entropy is viewed as one more fluid with velocity  $\mathbf{v}_s$ , chemical potential  $\mu_s = T$  and number density  $n_s = s$  where  $s$  is the entropy density. In this language, for instance, heat conduction translates to a velocity lag between the entropy fluid and the other fluids. For our purposes it makes sense to ignore conduction and assume that entropy is carried by the normal particles (electrons/muons), so that  $s = s_e + s_\mu$ . From the point of view of hydrodynamics this means that entropy and temperature should appear in the proton-electron-muon Euler equation.

The real importance of having a finite temperature in a neutron star core can be understood if we recall that the neutron/proton pairing energy is a bell-shaped function of  $T$  which means that, in certain regions, it may be exceeded by the thermal energy  $k_B T$ , thus leading to a local suppression of neutron superfluidity (and to a lesser extent of proton

superconductivity). As a result, the core may consist of adjacent multifluid and single fluid layers (the charged particle species are counted as a single fluid) of non-barotropic matter. Given that we have already discussed MHD equilibria in single-fluid stratified matter (Section 2.2), here we focus on the finite- $T$  multifluid regions (we should note, however, that the boundary physics between the aforementioned layers is poorly understood and well beyond the scope of this paper – our analysis may not apply in these boundaries).

The magnetic equilibrium in *npeμ* superfluid neutron stars is described, as before, by a pair of Euler equations. The neutron Euler equation is,

$$\rho_n \nabla (\tilde{\mu}_n + \Phi) = \mathbf{F}_n \Rightarrow \rho_n \nabla (\delta\tilde{\mu}_n + \delta\Phi) = \mathbf{F}_n. \quad (76)$$

The proton Euler contains most of the ‘new’ physics:

$$\rho_p \nabla (\tilde{\mu}_{pe} + \Phi) + \rho_p \frac{x_\mu}{x_p} \nabla (\tilde{\mu}_\mu - \tilde{\mu}_e) + s \nabla T = \mathbf{F}_p, \quad (77)$$

where we have used (74) and assumed  $m \gg m_\mu, m_e$ . Also, we have defined the muon fraction  $x_\mu = n_\mu/n$ , where  $n$  is the total particle number density. The background part of (77) leads to the beta equilibrium (75) in combination with a uniform temperature  $T$ . For the perturbed part, we have:

$$\rho_p \nabla (\delta\tilde{\mu}_{pe} + \delta\Phi) + \rho_p \frac{x_\mu}{x_p} \nabla \delta\gamma + s \nabla \delta T = \mathbf{F}_p, \quad (78)$$

where we have defined the electron-muon chemical difference

$$\delta\gamma \equiv \delta\tilde{\mu}_\mu - \delta\tilde{\mu}_e = \frac{1}{m} (\delta\mu_\mu - \delta\mu_e). \quad (79)$$

The total pressure is given by,

$$\nabla p = \rho_n \nabla \tilde{\mu}_n + \rho_p \nabla \tilde{\mu}_{pe} + m n_\mu \nabla \tilde{\mu}_\mu + s \nabla T, \quad (80)$$

and this leads to

$$\delta h = \delta\tilde{\mu}_n + x_p \delta\beta + x_\mu \delta\gamma + \tilde{s} \delta T, \quad (81)$$

where  $\tilde{s} = s/\rho$  is the specific entropy.

Following the procedure of the previous section we obtain the equivalent set of equations,

$$\nabla (\delta h + \delta\Phi) - \delta\beta \nabla x_p - \delta\gamma \nabla x_\mu - \delta T \nabla \tilde{s} = \frac{1}{\rho} (\mathbf{F}_p + \mathbf{F}_n), \quad (82)$$

$$\begin{aligned} & \nabla \left( \delta\beta + \frac{x_\mu}{x_p} \delta\gamma + \frac{\tilde{s}}{x_p} \delta T \right) - \delta\gamma \nabla \left( \frac{x_\mu}{x_p} \right) - \delta T \nabla \left( \frac{\tilde{s}}{x_p} \right) \\ &= \frac{1}{\rho_p} \mathbf{F}_p - \frac{1}{\rho_n} \mathbf{F}_n. \end{aligned} \quad (83)$$

The key new terms in these equations are the gradients of  $x_\mu$  and  $\tilde{s}$  and their ratios with  $x_p$ . Both terms are expected to be non-zero in realistic neutron stars and, between them, the muon composition gradient is expected to be the dominant effect (whenever muons are present) since the entropy term should scale with  $T/T_F \ll 1$ . In fact, the muon composition terms can be as large as the proton/neutron terms in the Euler equations (the relative magnitude of the muon-entropy terms can also be inferred from the *g*-mode calculations of Gusakov & Kantor (2013) and Passamonti et al. (2016)).

In principle, the superconducting magnetic forces will also be modified to some extent. For instance, it is well known that  $H_c$  becomes a function of the temperature (see Tinkham 1996). The variational derivation of these forces (Glampedakis et al. 2011) suggests that in addition to the

$\partial H_c / \partial \rho_{n,p}$  gradient terms we should also expect the presence of a similar  $\partial H_c / \partial T$  term. At the same time, given that the basic structure of the fluxtube array remains the same irrespective of the presence of muons and temperature, we expect that  $\mathbf{F}_p, \mathbf{F}_n$  will be given by expressions functionally similar to the ones of the previous section (in particular the  $\mathbf{F}_H$  part of the force should retain its form, Eqn. (72)).

In fact, for the point we wish to make here, we do not need to specify the exact form of these forces: as evident from Eqn. (83), the  $\delta\gamma \nabla(x_\mu/x_p)$  and  $\delta T \nabla(\tilde{s}/x_p)$  terms prevent the magnetic forces from being equal to a total gradient. In other words, we can conclude that the realistic hot superfluid  $npe\mu$  model does not admit a Grad-Shafranov-type equation. To what extent the system admits an arbitrary magnetic field is decided by the underlying symmetry.

### 3.2.1 Axisymmetric equilibria

Assuming an axisymmetric system we can show that, as was the case in the earlier single fluid non-barotropic model, the available equilibrium equations allow for an *arbitrarily* prescribed magnetic field and magnetic forces (the latter can only depend on  $\mathbf{B}$ ,  $H_c$  and background fluid parameters).

The model at hand has eight degrees of freedom<sup>3</sup> associated with the perturbed fluid, namely,  $\{\delta h, \delta\beta, \delta\gamma, \delta T\}$  and  $\{\delta s, \delta n_n, \delta n_p, \delta \tilde{\mu}_n\}$ . There are also eight equations available at our disposal: four of them come from the Euler equations (82), (83); three are ‘equations of state’ with a symbolic form  $\delta n_x = f_x(\delta\beta, \delta\gamma, \delta \tilde{\mu}_n, \delta T)$ ,  $x = \{n, p, s\}$ ; finally, the pressure equation (81) can be written as  $\delta \tilde{\mu}_n = f(\delta\beta, \delta\gamma, \delta T, \delta h)$ .

We can thus see that once the Euler equations have been solved with respect to the subset  $\{\delta h, \delta\beta, \delta\gamma, \delta T\}$  the remaining functions can be readily obtained algebraically. From the  $\theta$ -Euler equations we obtain

$$\delta h = \zeta + \mathcal{S}_p + \mathcal{S}_n, \quad (84)$$

$$\delta\beta = \tilde{\zeta} - \frac{x_\mu}{x_p} \delta\gamma - \frac{\tilde{s}}{x_p} \delta T + \mathcal{S}_p - \mathcal{S}_n, \quad (85)$$

where  $\zeta(r)$  and  $\tilde{\zeta}(r)$  are arbitrary functions and

$$\mathcal{S}_x(r, \theta) \equiv \frac{r}{\rho_x} \int d\theta F_x^\theta, \quad x = \{p, n\}. \quad (86)$$

Inserting these in the radial Euler components, we obtain an algebraic system for the remaining two functions:

$$\begin{aligned} \left(\frac{x_\mu}{x_p}\right)' \delta\gamma + \left(\frac{\tilde{s}}{x_p}\right)' \delta T &= \tilde{\zeta}' + \partial_r(\mathcal{S}_p - \mathcal{S}_n) \\ &\quad - \frac{1}{\rho_p} F_p^r + \frac{1}{\rho_n} F_n^r, \end{aligned} \quad (87)$$

$$\begin{aligned} \left(\frac{x_\mu}{x_p}\right)' \delta\gamma + \left(\frac{\tilde{s}}{x_p}\right)' \delta T &= \frac{1}{x_p} \left[ \zeta' - x_p' \tilde{\zeta} + \partial_r(x_n \mathcal{S}_n) \right. \\ &\quad \left. + x_p' \mathcal{S}_n + x_p \partial_r \mathcal{S}_p - \frac{1}{\rho} (F_p^r + F_n^r) \right]. \end{aligned} \quad (88)$$

The identical left-hand-sides mean that the system does not lead to a unique solution for  $\delta\gamma$  and  $\delta T$ . Moreover, from the

<sup>3</sup> Without counting the perturbed potential  $\delta\Phi$  which can be readily calculated from the Poisson equation once  $\delta\rho = \delta\rho_n + \delta\rho_p$  is known.

equality of the right-hand-sides we get a differential relation between the two gauge functions:

$$\zeta' - (x_p \tilde{\zeta})' = \frac{1}{\rho} \left(1 + \frac{\rho_p}{\rho_n}\right) F_n^r - \partial_r[(x_p + x_n) \mathcal{S}_n] \quad (89)$$

$$= \frac{1}{4\pi} (x_p + x_n)' B \frac{\partial H_c}{\partial \rho_n} = 0. \quad (90)$$

The last line of this equation was obtained using (66) and  $x_p + x_n = 1$  (note that the same result would hold for the case of normal protons, i.e. when  $\mathbf{F}_n = 0$ ).

The upshot of this discussion is that, in an axisymmetric neutron star composed of finite temperature  $npe\mu$  matter, there is enough freedom in the perturbed fluid parameters to balance arbitrarily specified magnetic forces. Qualitatively speaking, the system behaves as the single-fluid non-barotropic star of Section 2.2. Interestingly, this is also true with respect to the  $g$ -mode oscillations (see related comment at the end of Section 3.1): the presence of entropy and a muon component leads to the re-emergence of thermal/composition  $g$ -modes in superfluid neutron stars (Gusakov & Kantor 2013; Kantor & Gusakov 2014; Passamonti et al. 2016).

### 3.2.2 Non-axisymmetric equilibria

Moving on to general non-axisymmetric equilibria, we now need to take into account the presence of two additional Euler equations (the  $\varphi$ -components). With six available equations for the four functions  $\{\delta h, \delta\beta, \delta\gamma, \delta T\}$  we expect to have two constraints for the magnetic force components. Indeed, following the same procedure as in Section 2.2.2, we can combine the  $\theta$  and  $\varphi$  components to obtain:

$$\partial_\varphi(F_p^\theta + F_n^\theta) = \partial_\theta [\sin \theta (F_p^\varphi + F_n^\varphi)], \quad (91)$$

$$\partial_\varphi \left( F_p^\theta - \frac{\rho_p}{\rho_n} F_n^\theta \right) = \partial_\theta \left[ \sin \theta \left( F_p^\varphi - \frac{\rho_p}{\rho_n} F_n^\varphi \right) \right]. \quad (92)$$

From these we can produce the simpler relations:

$$\partial_\varphi(F_x^\theta) = \partial_\theta(\sin \theta F_x^\varphi), \quad x = \{p, n\}. \quad (93)$$

Using the superconducting forces (65) and (66) we find that the gradient terms are mutually cancelled out, leaving just one non-trivial constraint equation:

$$\partial_\varphi(F_H^\theta) = \partial_\theta(\sin \theta F_H^\varphi). \quad (94)$$

This result is clearly very similar to the one found in Section 2.2.2 and implies that the non-axisymmetric  $npe\mu$  system does not admit an arbitrary magnetic field equilibrium. We note that for non-superconducting protons we have  $\mathbf{F}_H \rightarrow \mathbf{F}_L$  and we recover exactly our earlier result (43).

Finally, for the fluid variables we obtain the same algebraic equations (84), (85) and (88) as in the axisymmetric case (with  $\zeta$  and  $\tilde{\zeta}$  now functions of  $r$  and  $\varphi$ ).

## 4 GENERAL RELATIVISTIC EQUILIBRIA

What we have learned so far about MHD equilibria in Newtonian stars can be carried over into the more realistic case of general relativistic stars. This section provides a ‘proof of principle’ analysis and as such it will suffice to consider the simple case of a single-fluid axisymmetric GRMHD system.



We show that (i) a barotropic EOS allows the formulation of a relativistic Grad-Shafranov equation and (ii) once more realistic models of matter are considered (with stratification and departure from chemical equilibrium), the Grad-Shafranov equation is lost and one is free to specify an arbitrary magnetic field equilibrium.

The stress-energy tensor for a perfect fluid coupled with an electromagnetic (EM) field is,

$$T^{\mu\nu} = T_F^{\mu\nu} + T_{\text{EM}}^{\mu\nu}, \quad (95)$$

where

$$T_F^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + pg^{\mu\nu}, \quad (96)$$

and

$$T_{\text{EM}}^{\mu\nu} = \frac{1}{4\pi} \left( g_{\kappa\lambda} F^{\mu\kappa} F^{\nu\lambda} - \frac{1}{4} g^{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda} \right). \quad (97)$$

We have used standard notation, with  $\epsilon$  denoting the energy density and  $u^\mu$  the local four-velocity of fluid elements. The (antisymmetric) Faraday tensor  $F_{\mu\nu}$  can be parametrised in terms of the four-potential  $A^\mu$ ,

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (98)$$

As usual, the introduction of  $A_\mu$  comes with the gauge freedom bonus,  $A_\mu \rightarrow A_\mu + \nabla_\mu f$ .

The Maxwell equations for the EM field are,

$$\nabla_\nu F^{\mu\nu} = 4\pi J^\mu, \quad \nabla_{[\alpha} F_{\beta\gamma]} = 0, \quad (99)$$

where  $J^\mu$  is the current density. An equivalent (and occasionally more practical) expression for the first equation is,

$$\partial_\nu (\sqrt{-g} F^{\mu\nu}) = 4\pi \sqrt{-g} J^\mu, \quad (100)$$

where  $g$  is the metric determinant. The Lorentz force is defined as,

$$F_L^\mu = -\nabla_\nu T_{\text{EM}}^{\mu\nu} = F^{\mu\nu} J_\nu, \quad (101)$$

where Eqns. (99) were used in the last step.

The full GRMHD equations of motion are given by  $\nabla_\nu T^{\mu\nu} = 0$ . Upon projecting orthogonally with respect to  $u^\mu$ , we obtain the Euler equation:

$$(\epsilon + p)u^\beta \nabla_\beta u_\alpha + \partial_\alpha p + u_\alpha u^\beta \partial_\beta p = F_{L\alpha}. \quad (102)$$

In addition, the MHD approximation is defined by the condition of a vanishing electric field,

$$E_\mu = F_{\mu\nu} u^\nu = 0. \quad (103)$$

#### 4.1 The relativistic Grad-Shafranov equation

In order to study magnetic equilibria in GRMHD we first assume a static and axisymmetric system. This implies a fluid four-velocity  $u^\mu = (u^t, 0, 0, 0)$  and a diagonal Weyl-type metric,

$$ds^2 = g_{tt} dt^2 + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\varphi\varphi} d\varphi^2, \quad (104)$$

with  $g_{\mu\nu} = g_{\mu\nu}(r, \theta)$ . One can treat the EM field as a perturbation of a spherical background star, implying the metric  $g_{\mu\nu} = \text{diag}[-e^{\nu(r)}, e^{\lambda(r)}, r^2, r^2 \sin^2 \theta]$  at leading order with respect to the field (see e.g. [Ciolfi et al. 2009](#)). However, the following derivation also applies using the metric (104).

Given these assumptions, the Euler equation (102) becomes,

$$F_{L\alpha} = \frac{1}{2}(\epsilon + p)g^{tt}\partial_\alpha g_{tt} + \partial_\alpha p. \quad (105)$$

From this it follows that the azimuthal force vanishes, i.e.  $F_{L\varphi} = 0$ . Then, the  $\varphi$ -component of the Lorentz force law (101) leads to

$$J^r \partial_r \Psi + J^\theta \partial_\theta \Psi = 0, \quad (106)$$

where we have defined the stream function<sup>4</sup>  $\Psi(r, \theta) \equiv A_\varphi$ .

The components  $J^r, J^\theta$  can be calculated from (100):

$$J^i = \frac{1}{4\pi\sqrt{-g}} \partial_j (\sqrt{-g} F^{ij}), \quad (107)$$

where  $i, j = \{r, \theta\}$  and  $i \neq j$ . Using this result in (106),

$$\partial_r \Psi \partial_\theta (\sqrt{-g} F^{r\theta}) - \partial_\theta \Psi \partial_r (\sqrt{-g} F^{r\theta}) = 0. \quad (108)$$

This implies

$$\sqrt{-g} F^{r\theta} = T(\Psi), \quad (109)$$

with  $T$  an arbitrary function representing the toroidal field degree of freedom.

From the remaining Maxwell equation components, after using  $F_{t\alpha} = 0$  and  $F_{\varphi\alpha} = -\partial_\alpha \Psi$ , we obtain  $J^t = 0$  and

$$J^\varphi = -\frac{1}{4\pi\sqrt{-g}} \partial_j \left( \frac{\sqrt{-g}}{g_{\varphi\varphi} g_{jj}} \partial_j \Psi \right), \quad (110)$$

where summation over  $j$  is assumed. We are now in a position to calculate the Lorentz force in terms of the stream functions. We find the trivial results  $F_{Lt} = F_{L\varphi} = 0$  and

$$F_{Li} = \left[ J^\varphi + \frac{TT'}{4\pi g_{tt} g_{\varphi\varphi}} \right] \partial_i \Psi \equiv \mathcal{A}_{\text{GR}} \partial_i \Psi, \quad (111)$$

where  $T' = dT/d\Psi$ . Using (110),

$$\mathcal{A}_{\text{GR}} = -\frac{1}{4\pi} \left[ \frac{\partial_j^2 \Psi}{g_{\varphi\varphi} g_{jj}} + \partial_j \left( \frac{\sqrt{-g}}{g_{\varphi\varphi} g_{jj}} \right) \frac{\partial_j \Psi}{\sqrt{-g}} - \frac{TT'}{g_{tt} g_{\varphi\varphi}} \right]. \quad (112)$$

The last step of this analysis is the manipulation of the Euler equation (105). This, however, entails making a choice for the EOS of matter.

#### 4.2 Barotropic matter

For a barotropic EOS we have relations  $p = p(\epsilon)$  and  $\epsilon = \epsilon(n)$  where  $n$  is the total baryon number density. From the first thermodynamical law we can obtain the Euler relation

$$p + \epsilon = \mu n, \quad \mu \equiv \frac{d\epsilon}{dn}, \quad (113)$$

where  $\mu$  is the chemical potential. These can be written in a differential form,

$$\partial_\alpha \epsilon = \mu \partial_\alpha n = \frac{p + \epsilon}{n} \partial_\alpha n. \quad (114)$$

<sup>4</sup> From the MHD condition (103) we have  $F_{\mu t} u^t = 0 \Rightarrow A_t = 0$ . Moreover, we can choose a gauge to make  $A_\theta = 0$ . The resulting EM potential is  $A_\mu = (0, A_r, 0, \Psi)$ .

For the pressure gradient we obtain,

$$\partial_\alpha p = -\partial_\alpha \epsilon + \partial_\alpha (\mu n) = (\epsilon + p) \frac{\partial_\alpha \mu}{\mu}. \quad (115)$$

Inserting this in the Euler equation,

$$F_{L\alpha} = (\epsilon + p) \partial_\alpha \left( \frac{1}{2} \nu + \log \mu \right) \equiv (\epsilon + p) \partial_\alpha X, \quad (116)$$

where we have used  $g_{tt} = -e^\nu$ .

After equating (116) and (111) we have,

$$\mathcal{A}_{GR} \partial_i \Psi = (\epsilon + p) \partial_i X. \quad (117)$$

This implies that  $X = X(\Psi)$  and

$$\mathcal{A}_{GR} = (\epsilon + p) M(\Psi) \quad (118)$$

with  $M$  arbitrary. Written explicitly, this result is:

$$\begin{aligned} \partial_j^2 \Psi + \partial_j \log \left( \frac{\sqrt{-g}}{g_{\varphi\varphi} g_{jj}} \right) \partial_j \Psi - \frac{g_{jj}}{g_{tt}} T T' &= \\ &= -4\pi(\epsilon + p) g_{\varphi\varphi} g_{jj} M(\Psi), \end{aligned} \quad (119)$$

with  $j = \{r, \theta\}$  summed. We have thus arrived at the desired relativistic Grad-Shafranov equation. This of course means that magnetic field equilibria in axisymmetric relativistic barotropic stars are not arbitrary. For actual calculations of relativistic Grad-Shafranov equilibria the reader can consult e.g. Colaiuda et al. (2008); Ciolfi et al. (2009).

### 4.3 Non-barotropic matter: restoring freedom in MHD equilibrium

A more sophisticated model for matter should account for the presence of stratification and deviations from chemical beta equilibrium. As already pointed out in previous sections, these effects require a non-barotropic EOS. As a more realistic benchmark model, in this section we assume a multi-constituent, single-fluid system (for a review see Andersson & Comer 2007). The constituents comprise neutrons, protons and electrons with number densities  $n_x$ ,  $x = \{n, p, e\}$ . Only two of these are independent since we always require local charge neutrality, i.e.  $n_p = n_e$ .

The upgraded EOS is of the form  $\epsilon = \epsilon(n_x)$  and from this we have,

$$d\epsilon = \sum_x \mu_x dn_x = \mu_n dn_n + \mu_{pe} dn_p, \quad (120)$$

where the chemical potentials are defined as  $\mu_x \equiv \partial\epsilon/\partial n_x$  and  $\mu_{pe} = \mu_p + \mu_e$ . We can write a similar expression with covariant/partial derivatives:

$$\partial_\alpha \epsilon = \mu_n \partial_\alpha n_n + \mu_{pe} \partial_\alpha n_p. \quad (121)$$

In terms of  $n = n_n + n_p$  and the proton fraction  $x_p = n_p/n$  this becomes,

$$\partial_\alpha \epsilon = \mu_n \partial_\alpha n + \beta \partial_\alpha (n x_p), \quad (122)$$

where  $\beta \equiv \mu_{pe} - \mu_n$  (not to be confused with the slightly different  $\beta$  parameter of the Newtonian models).

The total pressure of the system is given by,

$$p = -\epsilon + \sum_x n_x \mu_x = \epsilon + n \mu_n + n x_p \beta. \quad (123)$$

Taking the derivative of this and using (122) we find,

$$\begin{aligned} \partial_\alpha p &= n [\partial_\alpha (\mu_n + x_p \beta) - \beta \partial_\alpha x_p] \\ &= \frac{\epsilon + p}{\mu_n + x_p \beta} [\partial_\alpha (\mu_n + x_p \beta) - \beta \partial_\alpha x_p]. \end{aligned} \quad (124)$$

This result reduces to the barotropic expression (115) for (i) a system in beta equilibrium,  $\beta = 0$ , or (ii) a uniform composition,  $\partial_\alpha x_p = 0$  (in which case  $\mu = \mu_n + x_p \beta$ ).

Our earlier result (111) for the Lorentz force is valid irrespective of the EOS. However, the same is not true for the Euler equation (116). For the present non-barotropic model that equation is replaced by

$$F_{L\alpha} = (\epsilon + p) \left( \partial_\alpha \tilde{X} - \beta \partial_\alpha x_p \right), \quad (125)$$

where  $\tilde{X} \equiv \nu/2 + \log(\mu_n + x_p \beta)$ . Therefore,

$$\frac{\mathcal{A}_{GR}}{\epsilon + p} \partial_i \Psi = \partial_i \tilde{X} - \beta \partial_i x_p \Rightarrow \frac{\mathcal{A}_{GR}}{\epsilon + p} \partial_i \Psi \neq M(\Psi). \quad (126)$$

In other words, there is *no* Grad-Shafranov equation. It does not take much more work to show that for any arbitrary magnetic field  $\{\Psi, T(\Psi)\}$  the available fluid degrees of freedom can be chosen so that the equilibrium equations are satisfied. We have thus arrived at the same conclusion as in the case of Newtonian stars (Section 2.2).

## 5 CONCLUDING REMARKS

The allowed space of MHD equilibria in neutron stars is dependent on the nature of matter, namely the equation of state. In this paper, we have surveyed one's freedom to arbitrarily prescribe magnetic equilibria for different types of matter (i.e. number of distinct fluids, composition and entropy gradients, deviations from chemical equilibrium), degrees of symmetry (i.e. axisymmetry/non-axisymmetry) and types of gravity (i.e. Newtonian/general relativistic). This freedom depends on whether there are available fluid degrees of freedom to balance the magnetic force. Our results are summarised as follows (see Table 1 for a bird's eye view summary):

(i) Axisymmetric systems have a rich spectrum of arbitrariness with respect to MHD equilibria. We have found that the usual Grad-Shafranov equation is not only a property of simple barotropic stellar models. It can also control the MHD equilibrium in stratified matter provided the latter is multifluid (e.g. *npe* matter with neutron superfluidity). However, the addition of muons and entropy (hot *npeμ* matter) nullifies the Grad-Shafranov equation, eventually leading to freely specifiable magnetic fields (as in the case of single-fluid non-barotropic systems). Among other things, this freedom implies an arbitrary relative strength between the poloidal and toroidal field components.

(ii) In non-axisymmetric systems, the additional azimuthal components of the equations of motion prevent the magnetic field from being arbitrarily specified. The resulting constraint, at least for the case of non-superconducting matter, leads to a pair of Grad-Shafranov-like equations for the magnetic field's scalar degrees of freedom (i.e. the Euler potentials).

(iii) The transition from Newtonian to general relativistic gravity does not alter the above conclusions (but increases the complexity of the various equilibrium equations).

This paper has solely focused on MHD equilibria. The dynamical *stability* of these equilibria is a completely different and much harder question, with obvious repercussions for their astrophysical relevance. Recent work (Lander & Jones 2012) suggests that barotropic equilibria are generically unstable, but this may not be true for more realistic non-barotropic systems since the buoyancy force emerging in stratified matter is known to enhance stability (e.g., Akgün et al. 2013). Another avenue for instability could be provided by the interplay between rotation and magnetic field crust-core coupling during the initial spin down of newly formed neutron stars (Glampedakis & Lasky 2015).

Somewhat surprisingly, our work has some bearing on the nature of the Hall equilibrium in neutron star crusts. This equilibrium refers to the asymptotic  $t \rightarrow +\infty$  state of the magnetic induction equation when the field is sourced by electron currents (this is the so-called electron-MHD) and is set to evolve due to the Hall term, i.e.  $\partial_t \mathbf{B} = \nabla \times (\mathbf{v}_e \times \mathbf{B}) \propto \nabla \times \{(\mathbf{J}_e \times \mathbf{B})/n_e\}$ . We can immediately deduce that, in axisymmetry, the condition for Hall equilibrium is identical to Eqn. (13), thus leading to the Grad-Shafranov equation (17) with  $\rho \rightarrow \rho_e$  (Gourgouliatos & Cumming 2014). It is straightforward to generalise this result to the full non-axisymmetric case; we predict that the Hall equilibrium should be described by our 3-D Grad-Shafranov equation (24). It will be interesting to test this prediction with the recently developed numerical framework for 3-D Hall evolution in neutron star crusts (Gourgouliatos et al. 2016).

We hope that this paper will serve as a stepping stone for modelling the next-generation MHD equilibria in realistic neutron stars.

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